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Two three-state Potts models: the universality of confluent corrections to scaling

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Abstract. We present a study of an extended (35-power) low-temperature series for the susceptibility of the ordinary two-site three-state Potts model on the square lattice as well as reanalyses of several extant series for the three-site three-state Potts model on the triangular lattice. Dominant and confluent singularities are investigated and the exponents of both are found to be universal and in agreement with known and conjectured exact values.

In a recent series of papers (Adler *et al* 1982a, b, 1983) a new method of Padé analysis that explicitly accounts for confluent corrections to scaling has been developed. This method has been applied to the ordinary two-site three-state Potts model (Wu 1982) on the square lattice by Adler and Privman (1982, hereafter denoted as AP). For both the magnetisation (M) and the internal energy (E) partition function series of Enting (1980b, 1982, see table 2) the hypothesis of critical behaviour of the forms ($u=\exp(-J/kT)$)

$$M \sim (u_c-u)^{\beta}[1+a_{1M}(u_c-u)^{\Delta_1}+b_{1M}(u_c-u)+\ldots], \tag{1}$$

$$E \sim (u_c-u)^{1-\alpha}[1+a_{1E}(u_c-u)^{\Delta_1}+b_{1E}(u_c-u)+\ldots], \tag{2}$$

respectively has led to values of the dominant exponents β and α that are in agreement with known and conjectured exact (Wu 1982) results. See table 1 for details. The leading confluent exponent Δ_1 was found to be the same for both quantities, as predicted by universality (Wegner 1972). Similar β and α and Δ_1 estimates were found for the Hamiltonian series (Pearson 1980, 1982); however, the convergence of the γ range, namely $\gamma=1.449\pm0.027$, where γ is the critical exponent of the susceptibility χ,

$$\chi=f(u)\sim(u_c-u)^{-\gamma}[1+a_{1\chi}(u_c-u)^{\Delta_1}+b_{1\chi}(u_c-u)+\ldots], \tag{3}$$

is relatively wide due to the shortness of the series. Even so, this estimate is a substantial improvement when compared with ordinary Padé or with the high-temperature series estimate of Miyashita *et al* (1979) of $\gamma\approx1.50$ from a short (11-power) series without allowance for a confluent correction term. Previous to the analysis of AP, all the analyses of partition function series for the three-state Potts models had been problematical, and in particular the 18-power series of Enting (1980c) for the three-state three-site model on the triangular lattice appeared to exhibit slow

Table 1. Estimates of critical exponents.

Exponent	α	β	γ
Exact value (known or conjectured)	$1/3=0.333\ldots$	$1/9=0.1111\ldots$	$13/9=1.444\ldots$
Series results			
Partition function (two-site)			
$\Delta=1$	≈ 0.37	$\leqslant 0.110$	≈ 1.47
$\Delta\neq 1,\ b=0$	0.348 ± 0.008	0.1110 ± 0.0007	
	$\Delta_1=0.56\pm 0.14$	$\Delta_1=0.63\pm 0.19$	
$\Delta\neq 1,\ b\neq 0$			1.43 ± 0.04
		$b_{1M}=-(0.1\div 0.2)$	$\Delta_1=0.50\pm 0.24$
			$b_{1(x/u^4)}=2.2\pm 0.8$
Hamiltonian (two-site)			
$\Delta=1$	≈ 0.31	$\leqslant 0.1090$	$\gtrsim 1.49$
$\Delta\neq 1,\ b=0$	0.331 ± 0.009	0.1111 ± 0.0006	1.449 ± 0.027
	0.65 ± 0.12	$\Delta_1=0.54\pm 0.14$	$\Delta_1=0.53\pm 0.18$
$\Delta\neq 1,\ b\neq 0$		0.1113 ± 0.0012	
		$\Delta_1=0.54\pm 0.12$	
		$b_{1M}=-(0.1\div 0.2)$	
Partition function (three-site)			
$\Delta=1$	≈ 0.42	≈ 0.1105	
$\Delta\neq 1,\ b=0$	0.343 ± 0.020		
	$\Delta_1=0.4\pm 0.2$		
$\Delta\neq 1,\ b\neq 0$		0.1111 ± 0.0019	
		$\Delta_1=0.57\pm 0.05$	
		$b_{1M}\approx -4.5$	

convergence to the dominant exponents of the hard hexagon model (Baxter 1980), that Alexander (1975) suggested should lie in the three-state Potts model universality class. These exponents have since been conjectured (and partly proved) to apply to the three-state Potts model (Wu 1982 and references therein), but the correspondence between hard hexagon and Potts models does not seem to extend to corrections to scaling.

In the present work we study a new low-temperature partition function susceptibility series for the ordinary two-site three-state Potts model and reanalyse the extant three-site series; we demonstrate the universality of both the dominant and the confluent critical exponents, and show that the convergence of the dominant exponent is improved in all cases.

The low-temperature series were calculated using the techniques described by Enting (1978), with the partition function being constructed as a product of positive and negative powers of the partition functions of small finite lattices with fixed (spin-state 0) boundaries. The finite lattice partition functions are calculated by using a transfer matrix that adds one site at a time in the manner described by Enting (1980a). This computational refinement has made it possible to extend the series given by Enting (1980b) from u^{31} to u^{35} in the temperature variable u and from x^1 to x^2 in the field variable $x=1-\exp(-H/kT)$. The extension to x^2 enables us to calculate series for the zero-field susceptibility.

For finite lattices of width $\leqslant w$, the Potts model series can be obtained to order $4w+3$ in u. The state vectors on which the transfer matrices act are formally of dimension 3^w, but the symmetry between the 1 and 2 states means that only n_w of the components are distinct, where $n_w = 3n_{w-1} - 1$ and $n_1 = 2$. The use of the single-site transfer matrices means that the spatial reflection symmetry that would be present in a row-to-row formalism is lost, and so it is necessary to store the full $n_8 = 3281$ components of each of two vectors. (The elements of the transfer matrices are defined implicitly in a manner similar to that described by Enting (1978) and so no storage is required for the matrices.) The integer coefficients were calculated using the arithmetic of residues modulo the primes $p_i = 2^{15} - 19$, $2^{15} - 49$, $2^{15} - 51$ and $2^{15} - 55$. This means that the final calculation is correct modulo $P = \Pi_{i=1}^{4} p$. The regular behaviour of the coefficients indicates that four primes are sufficient to define the series given in table 2.

Table 2. New coefficients of the low-temperature series expansions $Z = \Sigma_n a_n u^n$, $1 - M = \Sigma_n b_n u^n$, $\chi = \Sigma_n c_n u^n$, for the two-site model. The coefficients a_n and b_n for $n \leq 31$ appear in Enting (1980a).

n	c_n	n	a_n	b_n	c_n
0	0	18			3 201 728
1	0	19			8 670 688
2	0	20			25 713 154
3	0	21			67 206 560
4	2	22			203 077 760
5	0	23			532 881 432
6	16	24			1 558 159 918
7	16	25			4 250 639 632
8	100	26			11 956 293 152
9	216	27			33 296 697 848
10	844	28			92 820 406 096
11	1 552	29			257 249 275 776
12	7 844	30			721 023 458 656
13	12 112	31			1 986 080 278 600
14	60 268	32	4 847 112 666	247 542 929 499	5 561 045 323 298
15	118 944	33	11 876 028 924	648 347 258 796	15 359 165 767 512
16	424 072	34	29 820 747 120	1 713 912 378 552	42 717 426 328 784
17	1 081 392	35	76 592 341 404	4 559 593 914 288	118 457 421 095 792

The new susceptibility series for the two-site model as well as the partition function, magnetisation and susceptibility series for the three-site model have been analysed with the techniques developed by Adler *et al* (1982a, b). These techniques, which are discussed at length in Adler *et al* (1983), employ the Roskies (1981) map from u to $y = 1 - (1 - u/u_c)^{\Delta_1}$, in order to suppress the influence of the non-analytic correction term in the critical behaviour in, for example, the susceptibility. The non-analytic correction term which has the coefficient $a_{1\chi}$ in (3) has been shown (see AP) to be responsible for systematic errors in earlier Padé analyses of Potts model series. This can be demonstrated by considering the Dlog Padé evaluation of γ via

$$K(u) = (u_c - u)\, \mathrm{d} \ln f(u)/\mathrm{d}u = \gamma - \frac{a_{1\chi}\Delta_1(u_c - u)^{\Delta_1} + b_{1\chi}(u_c - u) + \ldots}{1 + a_{1\chi}(u_c - u)^{\Delta_1} + b_{1\chi}(u_c - u) + \ldots} \tag{4}$$

where $K(u) \simeq \Sigma_{n=0}^{N} K_n u^n$ is derived from a finite number of terms in the series for $\chi(u)$, and taking different $[L, M]$ Padé approximants to $K(u)$ at u_c. The non-analytic terms in (4) may lead to systematic errors in the evaluation of the exponent γ since Padé approximants do not generally converge in the presence of branch cuts.

To avoid this problem, we invoke the Roskies (1981) transformation, replacing Δ_1 with a variable Δ, and search for the region at $\Delta \sim \Delta_1$, where the non-analyticity of the leading confluent term will be suppressed. We study Padé approximants to the function

$$G_\Delta(y) = \Delta(y-1)\, \mathrm{d}(\ln \chi(y))/\mathrm{d}y = \gamma - \frac{\tilde{a}_{1\chi}\Delta_1(y-1)^{\Delta_1/\Delta} + \ldots}{1 + \tilde{a}_{1\chi}(y-1)^{\Delta_1/\Delta} + \ldots}$$

where $\tilde{a}_{1\chi} = a_{1\chi}T_c^{\Delta_1}$, and plot graphs of γ as a function of Δ. When $u = u_c$, $y = 1$ and for each $[L, M]$ approximant we have a curve

$$G_\Delta^{[L,M]}(y=1) = \gamma_{\text{out}}(\Delta).$$

When $\Delta = 1$, γ_{out} $(\Delta = 1)$ will be the result obtained from a usual Dlog Padé analysis and when Δ is near the correct Δ_1 we may linearise $G_\Delta(y)$ in the difference $\Delta - \Delta_1$ and obtain

$$G_\Delta(y) \approx \gamma + \tilde{a}_{1\chi}(1-y)\ln(1-y)(\Delta - \Delta_1) + \ldots$$

where the corrections will now change the slopes of the curves $\gamma = \gamma_{\text{out}}(\Delta)$ in the (Δ, γ) plane. If there were no analytic $(b_{1\chi})$ or higher-order confluent terms all the curves would intersect at the correct (Δ_1, γ); these other corrections smear the convergence somewhat and we have an intersection region near this point. We read off values for the dominant exponent γ and the correction exponent Δ_1 from the graph using the somewhat subjective intersection region to determine error bounds and may compare these with the results of a usual Dlog Padé by looking along the line $(1, \gamma)$.

We consider the new susceptibility series first. In figure 1, we present curves for the series $\chi(u)/u^4$ for the square lattice two-site model (where u_c is known exactly from duality) and observe that there is a clear, but erroneous convergence at $\Delta \simeq 1$. The conjectured value $\gamma = 13/9 = 1.444\ldots$ and the Δ_1 (series) $= 0.57 \pm 0.13$ from AP are illustrated by a bar and we can see that most of the Padé curves cross the bar, although the strong analytic $b_{1\chi}$ term apparently destroys the intersection region. The hypothesis that the analytic term destroys the intersection region was investigated by studying (Aharony 1982) $(\chi(u)/u^4)/[1 + b(u_c - u)]$ for various values of b, and for $b = +2.2$ the convergence region near $\Delta \sim 1$ disappears. We present the curves for $b = +2.2$ in figure 2 and from this and similar plots conclude that $b_{1(\chi/u^4)} = 2.2 \pm 0.8$. In these plots an apparent 'convergence' develops for a lower Δ and we estimate

$$\gamma = 1.43 \pm 0.04, \qquad \Delta_1 = 0.50 \pm 0.24,$$

consistent with the conjecture for γ and Δ_1(series) of AP.

We now turn to the extant three-site triangular lattice series. These are somewhat shorter (18-power), but extremely interesting in view of the question of universality of correction to scaling terms, which does not seem to occur between two-state Potts and hard hexagon models. The possible universality between corrections to scaling for the two- and three-site Potts models was suggested by Enting (1980c) on the grounds that in both models the usual Dlog Padé exponent estimates erred from the hard hexagon exponents in the same direction. Just as for the two-site model on the square lattice, u_c is known exactly from duality arguments (Baxter *et al* 1978) and

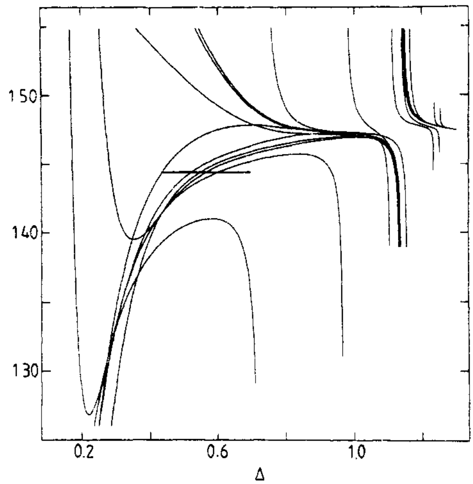

Figure 1. $\gamma(\Delta)$ curves for the $\chi(u)/u^4$ series derived from the susceptibility series of the two-site Potts model on the square lattice (table 2) obtained using [13, 17], [14, 16], [15, 15], [16, 14], [17, 13], [13, 16], [14, 15] and [16, 13] Padé approximants. The bar indicates γ(conj) and Δ_1(series) for AP.

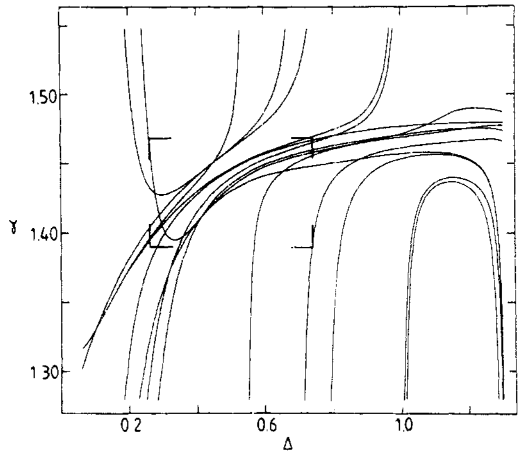

Figure 2. $\gamma(\Delta)$ curves for the series $(\chi(u)/u^4)/[1+b(u_c-u)]$ of the two-site Potts model on the square lattice with $b=+2.2$, obtained using the same Padé approximants as in figure 1.

we define the magnetisation

$$M = 1 - \tfrac{3}{2}(\partial Z/\partial x)/Z,$$

susceptibility

$$\chi = \frac{\partial^2 Z}{\partial x^2}\Big/ Z - \frac{\partial Z}{\partial x}\Big/ Z - \left(\frac{\partial Z}{\partial x}\Big/ Z\right)^2$$

and energy per site E where

$$E/J = u(\partial Z/\partial u)/Z$$

after Enting (1980c) where Z is the partition function. The critical value of $E/J = \frac{1}{3}$ and $u_c = \frac{1}{4}$.

The magnetisation series results are presented in figure 3. Again there is a clear convergence region at $\Delta \sim 1$ and we indicate the line of conjectured $\beta = 0.11111\ldots$ and Δ_1(series) from AP with a bar. We again attribute the lack of convergence to a 'strong' b_{1M} term and on studying series for $M/[1+b(u_c-u)]$ find that the apparent convergence near $\Delta = 1$ is weakened for $4^+ < -b \leq 6.0$. When $b \simeq -4.5$, the $\Delta \simeq 1$ region nearly disappears (see figure 4 for $b = -4.5$), and we find

$$\beta = 0.1111 \pm 0.0019, \qquad \Delta = 0.57 \pm 0.05.$$

It is not clear whether this b represents an estimate of b_{1M}.

In figure 5 we present the energy series. Here we may have two intersection regions for $b=0$, or perhaps a single, rather wide region, since division by $1+b(u_c-u)$ for a large range of $b \gtrless 0$ values does not appreciably affect the structure near $\Delta \approx 0.75$. Assuming the former situation, we find

$$\alpha - 1 = -0.657 \pm 0.020, \qquad \Delta_1 = 0.4 \pm 0.2,$$

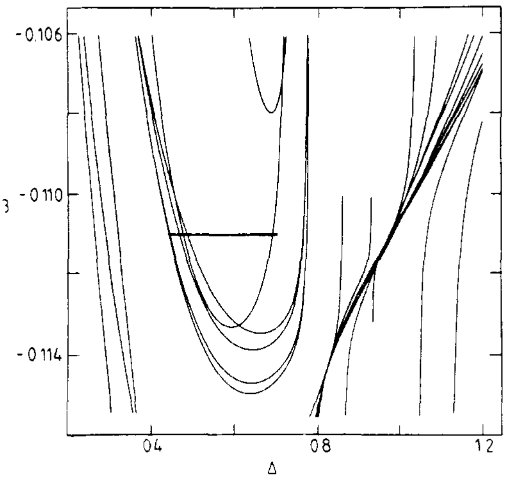

Figure 3. $-\beta(\Delta)$ curves for the three-site Potts model on the triangular lattice derived from the M series of Enting (1980c) using [7, 10], [8, 9], [9, 8], [10, 7], [7, 9], [8, 8], [9, 7], [7, 8] and [8, 7] Padé approximants. The bar indicates β(conj) and Δ_1(series) from AP.

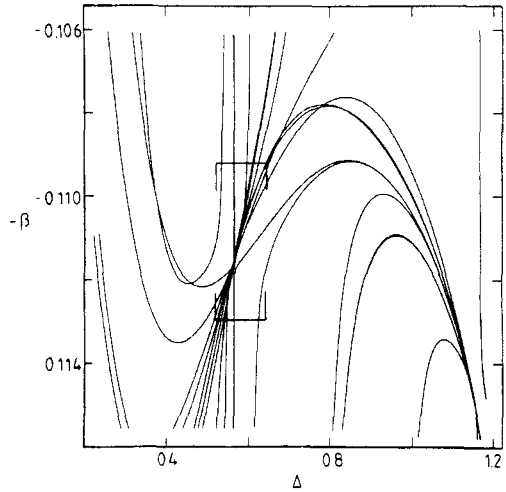

Figure 4. $-\beta(\Delta)$ curves for the series $M/[1+b(u_c-u)]$ for the three-site model on the triangular lattice where $b=-4.5$ obtained using the same Padé approximants as in figure 3.

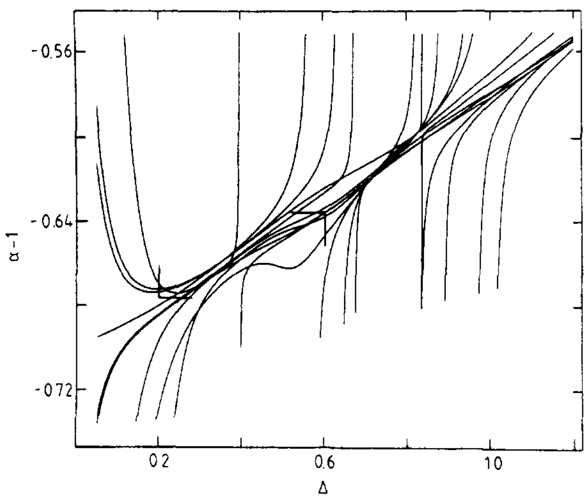

Figure 5. $\alpha(\Delta)-1$ curves for the $E_s/J=\frac{1}{3}-E/J$ series derived from the Z series of Enting (1980c) obtained using the same Padé approximants as in figure 3. The box assumes that the structure near $\Delta\sim 0.7$ is a second convergence region; this may not be so.

which is a large improvement on the usual Dlog Padé value of $\alpha-1>-0.6$. We also studied a $\chi(u)/u^3$ series for this model, but for input $|b|\leq 6$ the only intersection region was at $\Delta\sim 1$, although results consistent with γ(conj) and Δ_1(series) were found.

On comparison of figures 3 and 5 (this paper) with figures 2 and 3, and 5, respectively, of AP we observe a strong universality in the overall tendencies. This comparison is rather stronger when we compare figures 3 and 5 with our unpublished analyses of shorter magnetisation and energy partition function series for the two-site model. From the estimates of Δ_1 and these patterns we can confirm Enting's (1980c) suggested universality of correction to scaling effects for their two models, but we must caution that, just as neglect of the a_1 terms in both models led to similar errors in evaluating dominant exponents, we should be aware of the possibility of similar systematic errors in the Δ_1 estimates in both models. In particular, in both the three-site magnetisation partition function series (figures 3 and 4 above) and the usual two-state magnetisation Hamiltonian (Pearson 1980, 1982) series presented in figures 3 and 4 of AP, we observe that division by a $1+b(u_c-u)$ term tends to sharpen the Δ_1 estimate. We feel this may be spurious, and note that large uncertainties in confluent exponent series estimates appear to be usual (Adler *et al* 1982a, b, 1983, Chen *et al* 1982).

The results of the above calculations as well as those presented in AP are summarised in table 1, together with some exact and conjectured results for comparison purposes. We can make some general conclusions about relative amplitudes of confluent correction terms in these models, for example the ratio of the amplitude of the analytic term to that of the non-analytic term appears to be $\gg 1$ for susceptibility series, ~ 1 for magnetisation series and $\ll 1$ for internal energy series.

Overall agreement between our dominant and confluent exponent estimates is seen to be excellent and the universality of both for Potts models is clearly established.

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